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An Algorithm for a Special Class of Generalized Transportation-Type Problems

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An Algorithm for a Special Class of Generalized Transportation-Type Problems

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FOREWORD

This paper describes an algorithm developed for solving a special class of generalized transportation-type problems of moderate size. Problems concerned with optimal allocations of resources subject to meeting a given set of requirements such as marketing, routing, production, and weapons allocation are frequently of the generalized transportation type.

The generalized transportation-type problem considered here is a linear programming problem with solutions giving the allocations x_{ij} of the jth resource to the ith operation such as to maximize a given profit function. The requirements specify the limits on each of the n available resources as well as the operational limits of each of the m operations. In addition the operational capacity of the ith operation when the jth resource is assigned to it is known. The structure of such problems (one constraint for each row i and each column j) enables an algorithm more efficient than the general simplex algorithm to be used for finding a solution.

The algorithm is intended to solve moderate-sized problems faster than will general simplex algorithms. It requires less computer storage than general simplex algorithms, thus making it particularly useful when a limited-capacity computer memory is all that is available.

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ABSTRACT

The algorithm described in this paper is used to solve a special class of linear programming problems characterized by constraint coefficient matrices having generalized transportation structure. Specifically, n available resources are allocated to m capacity-limited operations (where the operational capability of assigning the jth resource to the ith operation is known) such as to maximize the total profit for the system. The row-and-column structure of such problems permits an algorithm more efficient than the general simplex algorithm to be used to solve moderate-sized problems (problems where loop-tracing techniques or equivalent schemes are not required). It is not required in the problem statement that all the resources be allocated or that all operations be performed to capacity limits. It is characteristic of such problems, however, that the optimizing solution usually requires that at least one of the two conditions holds, i.e., either supply or demand is exhausted. The paper contains a description of the algorithm, a computer program, an example illustrating its application, and some comparisons with the general simplex algorithm in solving the same problem.

1. INTRODUCTION

The algorithm presented here yields optimal solutions to a special class of linear programming problems that are characterized by constraint coefficient matrices having generalized transportation structure.† The algorithm preserves primal feasibility and the complementary slackness condition at all times; hence feasibility of the dual constraints forms a set of necessary and sufficient conditions for testing optimality.

The need for the present algorithm arose initially in application to an optimal weapons-allocation problem as part of a larger nonlinear minimax problem employed in an earlier RAC study[1] in this area.

A specialized algorithm (similar to the one given here) for generalized transportation-type problems appears to have been first used by Ferguson and Dantzig.[2,3]

The algorithm can be divided computationally into two phases: (1) the matrix maximum phase and (2) the simplex phase. In phase 1 the algorithm permits only vectors associated with positive cost to enter the basis and only basis vectors associated with slack variables to leave the basis. In phase 2 the selection of the next neighboring vertex is currently made as it is done in most simplex algorithms (see Ref 4, Lecture V and the appendix).

The particular structure of the constraint coefficient matrix permits economy of computation by employing the equivalent of a doubly indexed simplex algorithm.

2. PROBLEM STATEMENT

The algorithm presented in subsequent sections yields an optimal solution to the following class of linear programming problems.

†The details of this structure will be considered in Sec 3, "Problem Structure."

Maximize

$$\sum_{i,j}^{m,n} c'_{ij} x'_{ij} \qquad \text{with respect to} \qquad x'_{ij}$$

subject to the constraints

$$\begin{aligned}
&\sum_{j=1}^{n} d'_{ij} x'_{ij} \le a_i \ , \ a_i > 0 \ ; \ (i = 1, \ldots, m) \\
&\sum_{i=1}^{m} h'_{ij} x'_{ij} \le b_j \ , \ b_j > 0 \ ; \ (j = 1, \ldots, n) \\
&\left.\begin{aligned} x'_{ij} &\ge 0 \\ c'_{ij} &\ge 0 \\ d'_{ij}, h'_{ij} &> 0 \end{aligned}\right\} \begin{matrix} (i = 1, \ldots, m) \\ (j = 1, \ldots, n) \end{matrix}
\end{aligned} \tag{1}$$

Under the correspondences

$$\begin{aligned}
x_{ij} &= h'_{ij} x'_{ij} \\
d_{ij} &= d'_{ij}/h'_{ij} \\
c_{ij} &= c'_{ij}/h'_{ij}
\end{aligned} \tag{2}$$

an optimal solution to Prob 1 can be found from finding an optimal solution to Prob 3.

Maximize

$$\sum_{i,j}^{m,n} c_{ij} x_{ij} \qquad \text{with respect to} \qquad x_{ij}$$

subject to the constraints

$$\begin{aligned}
&\sum_{j=1}^{n} d_{ij} x_{ij} \le a_i \ , \ a_i > 0 \ ; \ (i = 1, \ldots, m) \\
&\sum_{i=1}^{m} x_{ij} \le b_j \ , \ b_j > 0 \ ; \ (j = 1, \ldots, n) \\
&\left.\begin{aligned} x_{ij} &\ge 0 \\ c_{ij} &\ge 0 \\ d_{ij} &> 0 \end{aligned}\right\} \begin{matrix} (i = 1, \ldots, m) \\ (j = 1, \ldots, n) \end{matrix}
\end{aligned} \tag{3}$$

The algorithm finds an optimal solution to Prob 3.

It should be observed that both the row and column constraints are inequalities. It is characteristic of such problems that the optimizing solution has the property that either all the row constraints, or all the column constraints, or both all row and all column constraints are binding when all $c_{ij} > 0$. If equalities are imposed on the column constraints and the row inequalities are of either type, we have the generalized transportation problem considered by Hadley.[5]

If both row and column constraints are equalities, $\Sigma a_i = \Sigma b_j$, and $d_{ij} = 1$ for all i, j, the problem reduces to the standard transportation problem.

3. PROBLEM STRUCTURE: GENERAL DISCUSSION

The general simplex algorithm may be used to solve Probs 1 or 3. For large m and n, however, it is not practical to do so. Writing the components x_{ij} of $x \in E^{mn}$ using a single component subscript index k for x_k (as is done when using the general simplex algorithm), we see that the constraint matrix A contains $mn(m + n - 2)$ zeros.

If g is the component subscript indexing function $[g(i, j) = k]$ for the vector x, then for the problem

$$\begin{array}{lll} \max_x \langle c,x\rangle & x \in E^{mn} & c \in E^{mn} \\ Ax \leq b & b \in E^{m+n} & b > 0 \\ x \geq 0 & & \end{array} \tag{4}$$

A assumes either of the two structures

$$A = \left(\begin{array}{c|c|c|c} d_{11} \cdots d_{1n} & d_{21} \cdots d_{2n} & \cdots & d_{m1} \cdots d_{mn} \\ \hline I_n & I_n & I_n & I_n \end{array}\right) \tag{5}$$

when $g(i,j) = n(i-1) + j = k \; (1 \leq i \leq m, \; 1 \leq j \leq n)$

where I_n is the identity matrix of order n or

$$A = \left(\begin{array}{c|c|c|c} \mathbf{1}_m & \mathbf{1}_m & \cdots & \mathbf{1}_m \\ \hline \begin{matrix} d_{11} & & \\ & d_{21} & \\ & & \ddots \\ & & & d_{mn} \end{matrix} & \begin{matrix} d_{12} & & \\ & d_{22} & \\ & & \ddots \\ & & & d_{m2} \end{matrix} & \ddots & \begin{matrix} d_{1n} & & \\ & \ddots & \\ & & d_{mn} \end{matrix} \end{array}\right) \tag{6}$$

when $g(i,j) = m(j-1) + i = k \; (1 \leq j \leq n, \; 1 \leq i \leq m)$

where $\mathbf{1}_m$ is a row vector of m ones.

†The right-hand side column vector b here includes $m + n$ components (m a_i and n b_j) as in Prob 3.

Any other indexing by g (besides interchanging upper and lower blocks) produces a less uniform structure for A. Structure 5 for A is associated with generalized transportation-type problems. When all $d_{ij} = 1$ the structure of A in Structure 5 is that of coefficient matrices associated with transportation problems. Structure 5 for A will be assumed when introducing suitable basis vectors for the solution space later on.

4. OPTIMALITY CRITERIA

The algorithm preserves primal feasibility P and the complementary slackness condition S at all times and uses the feasibility of the dual-programming problem constraints D as the optimality test criterion.

The three sets of Conditions P, D, and S are explicitly

$$P: \begin{cases} Ax + I_{m+n} x_s = b \\ x \geq 0 \\ x_s \geq 0 \end{cases} \qquad D: \begin{cases} A'w - I_{mn} w_s = c \\ w_s \geq 0 \\ w \geq 0 \end{cases} \tag{7}$$

$$S: \langle w, x_s \rangle + \langle w_s, x \rangle = 0$$

where $'$ denotes transposition
$\langle \, , \rangle$ denotes inner product
$b \geq 0$
$c > 0$

See Ref 6, Pt 2, p 58, for a discussion of Conditions 7.

Real vectors x, w that satisfy Conditions P, D, and S also solve the pair of dual linear programming problems

$$\max_x \langle c, x \rangle \quad \text{subject to P} \tag{8a}$$

$$\min_w \langle w, b \rangle \quad \text{subject to D} \tag{8b}$$

Problem 8a is solved with Conditions P and S always holding, hence Condition D becomes the set of necessary and sufficient conditions for optimality.

In practice the algorithm enforces the following stronger form $\bar{S}$ of Condition S, namely,

$$\langle w, x_s \rangle + \langle w_s, x \rangle = 0 \quad \underline{\text{at the component level,}} \tag{7$\bar{S}$}$$

i.e., $x_{ij} > 0 \Rightarrow w_{s_{ij}} = 0$; $x_{s_i} > 0 \Rightarrow w_i = 0$. Since the dual space constraint $A'w - I_{mn}w_s = c$ holds for all w, w_s throughout the algorithm, Conditions $7\bar{D}$ become the set of necessary and sufficient conditions for optimality, i.e.,

$$w_s \geq 0 \ , \ w \geq 0 \qquad (7\bar{D})$$

5. PROCESSES OF THE ALGORITHM

The general processes of the algorithm, the details of which will follow, are

(1) Generate basic primal feasible solution using complete or partial matrix maximum (Conditions 7P are satisfied by exactly $m + n$ positive primal variables x, x_s while the primal objective function is increased).

(2) Solve for the dual space variables u_i and v_j using Condition $7\bar{S}$ and knowledge of the structure of basis vectors associated with the positive primal variables.

(3) Perform optimality test (test Conditions $7\bar{D}$). If there are no violations the current basic primal solution is optimal.

(4) For nonoptimal solutions find the largest violation of Conditions $7\bar{D}$. Identify the associated vector for entry into the basis for the primal solution space.

(5) Find the representation of the entering vector in terms of vectors in current basis.

(6) Preserving primal feasibility Conditions P, select vector to leave the current basis.

(7) Express the solution in terms of the new basis.

(8) Return to step 2.

6. DETAILS OF THE ALGORITHM

The detailed description of the steps of the algorithm is presented here.

1. Generation of Basic Primal Feasible Solutions

The matrix maximum method of generating solutions x_{ij} is a process that makes allocations (assigns values to x_{ij}) to payoff elements c_{ij} of a matrix $\bar{P}$ of payoffs as follows:

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Let

$$\bar{P} = \{c_{ij} | x_{in'} \neq 0 \text{ and } x_{m'j} \neq 0\} \qquad \begin{matrix} (i = 1, \dots, n) \\ (j = 1, \dots, m) \\ n' = n + 1 \\ m' = m + 1 \end{matrix} \tag{9}$$

where

$$x_{in'} = a_i - \sum_{j=1}^{n} d_{ij} x_{ij}$$

$$x_{m'j} = b_j - \sum_{i=1}^{m} x_{ij}$$

$x_{in'}$ and $x_{m'j}$ represent the residual "slack" in row i and column j (Prob 3) after a set $\{x_{ij}\}$ of allocations has been made. In Conditions 7 x_s = $(x_{1n'}, \dots, x_{mn'}, x_{m'1}, \dots, x_{m'n})'$. If either $x_{in'}$ or $x_{m'j}$ = 0 then no further allocations involving row i or column j can be made since either the ith capacity has been achieved or the jth resource exhausted. Initially $\bar{P}$ is the matrix of all c_{ij} since $x_{in'} = a_i$ and $x_{m'j} = b_j$.

For each allocation x_{ij} of the matrix maximum method let

$$c_{k\ell} = \max_{ij} \{c_{ij} | c_{ij} \in \bar{P}\} \tag{10}$$

then choose

$$\begin{aligned} x_{k\ell} &= \min \left\{ \frac{a_k - \sum_{j=1}^{m} d_{kj} x_{kj}}{d_{k\ell}} \; ; \; b_\ell - \sum_{i=1}^{n} x_{i\ell} \right\} \\ &= \min \left\{ \frac{x_{km'}}{d_{k\ell}} \; ; \; x_{m'\ell} \right\} \end{aligned} \tag{11}$$

This choice for the value assigned to $x_{k\ell}$ eliminates either row k or column ℓ from the matrix $\bar{P}$ of payoffs for the next iteration. The new values $x'_{kn'}$, $x'_{m'\ell}$ for $x_{kn'}$ and $x_{m'\ell}$ are found as follows:

If

$$x_{k\ell} = \frac{x_{kn'}}{d_{k\ell}}$$

then

$$\begin{aligned} x'_{kn'} &= x_{kn'} - d_{k\ell} x_{k\ell} = 0 \\ x'_{m'\ell} &= x_{m'\ell} - x_{k\ell} \end{aligned} \tag{12}$$

If

$$x_{k\ell} = x_{m'\ell}$$

then (13)

$$x'_{kn'} = x_{kn'} - d_{k\ell}x_{m'\ell}$$

$$x'_{m'\ell} = x_{m'\ell} - x_{k\ell} = 0$$

In the first case $\overline{P}_{new} = \overline{P} - \{c_{kj} | x'_{kn'} = 0\}$ $(j = 1, \ldots, n)$. In the second case $\overline{P}_{new} = \overline{P} - \{c_{i\ell} | x'_{m'\ell} = 0\}$ $(i = 1, \ldots, m)$.

If $\frac{x_{kn'}}{d_{k\ell}} = x_{m'\ell}$ then an arbitrary decision is made to perturb $x_{kn'}$ by a small amount epsilon.

The matrix maximum procedure can be terminated in either of two ways, by exhausting the matrix $\overline{P}$ of payoffs (complete matrix maximum) or by assigning a fixed number (less than the number of iterations required to exhaust the matrix of payoffs) of positive allocations x_{ij} to be made (partial matrix maximum).

Throughout the matrix maximum iterations exactly $m + n$ elements of the vector $\overline{X}$ of allocations $(x_{ij}, x_{in'}, x_{m'j})$ are positive. The vector $\overline{X}$ satisfies Conditions 7P for primal feasibility.

The matrix maximum procedure proceeds from a vertex of the solution space to a neighboring vertex as does the simplex procedure, but specifically it proceeds to the vertex that has one less positive slack component and one more positive nonslack component (i.e., component having positive cost c_{ij}); hence the former is more efficient using a per iteration comparison. The matrix maximum procedure is not sufficient, however, to achieve optimality in general.

2. Solving for Dual Space Variables

The vector $\overline{X}$, resulting from application of the matrix maximum process (partial or complete), is a candidate optimizing point since it is an extreme point (Ref 4, p 58) of the convex set K of points $(x, x_s)'$ satisfying Conditions 7P

$$Ax + I_{m+n}x_s = b, \quad x \geq 0, \ x_s \geq 0 \tag{7P}$$

The linearly independent set (a basis) of $m + n$ vectors corresponding to $X^{\circ} = (x^{\circ}, x_s^{\circ})'$ (the subvector of positive components of $\bar{X}$) is defined as follows:

$$
\begin{array}{lll}
\text{If } x_{ij} > 0 \text{ then } d_{ij}\vec{e}_i + \vec{e}_{m+j}{}^{\dagger} & \text{is a member of the basis} & \\
\text{If } x_{in'} > 0 \text{ then } \vec{e}_i & \text{is a member of the basis} & \qquad (14) \\
\text{If } x_{m'j} > 0 \text{ then } \vec{e}_{m+j} & \text{is a member of the basis} &
\end{array}
$$

Recall that in the matrix maximum process if $x_{ij} > 0$ then not both $x_{in'} > 0$ and $x_{m'j} > 0$; hence if $d_{ij}\vec{e}_i + \vec{e}_{m+j}$ is a basis vector then not both $\vec{e}_i$ and $\vec{e}_{m+j}$ are basis vectors. Conversely, if both $x_{in'} > 0$ and $x_{m'j} > 0$ then $x_{ij} = 0$; hence if $\vec{e}_i$ and $\vec{e}_{m+j}$ are basis vectors then $d_{ij}\vec{e}_i + \vec{e}_{m+j}$ is not. The set of $m + n$ column vectors selected from the matrix (A, I_{m+n}) using Definition 14 and denoted by B (the ordered matrix of such column vectors) is thus linearly independent and satisfies the condition

$$BX^{\circ} = b$$

Hence $\bar{X}$ is an extreme point of K.

Corresponding to X° satisfying the equation $BX^{\circ} = b$ is a vector w° satisfying the equation $B'w^{\circ} = c^{\circ}$ where c° is the vector of costs (payoffs) associated with X°. If the $m + n$ components of w are written $w = (u_1, \ldots, u_m, v_1, \ldots, v_n)'$ then the scalar form of the equation $B'w = c$ or $w'B = c'$ is

$$
\left.
\begin{array}{lll}
d_{ij}u_i + v_j = c_{ij}^{\circ} & \text{if} & x_{ij} > 0 \\
u_i = 0 & \text{if} & x_{in'} > 0 \\
v_j = 0 & \text{if} & x_{m'j} > 0
\end{array}
\right\} (m+n) \text{ equations} \qquad (15)
$$

Solutions to Eqs 15 satisfy Condition 7S, $\langle w, x_s \rangle + \langle w_s, x \rangle = 0$. Since $x_{in'} > 0$ implies $u_i = 0$ and $x_{m'j} > 0$ implies $v_j = 0$ then $\langle w, x_s \rangle = 0$. Similarly, if $x_{ij} > 0$ implies $w_{s_{ij}} = d_{ij}u_i + v_j - c_{ij} = 0$ then $\langle w_s, x \rangle = 0$.

3. Optimality Test (Testing Conditions 7$\bar{D}$)

The set of necessary and sufficient Conditions 7$\bar{D}$ required for optimality of X° is rewritten here for reference.

$^{\dagger}\vec{e}_i$ here is a unit column vector in E^{m+n}.

$\vec{e}_i = (0, 0, \ldots, 1, 0, \ldots, 0)'$.
ith component

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$$A'w - I_{mn}w_s = c$$

$$\left.\begin{aligned} w_s &\leq 0 \\ w &\geq 0 \end{aligned}\right\} \;(7\bar{D}) \qquad (7D)$$

The scalar form of Conditions 7D is

$$d_{ij}u_i + v_j - w_{s_{ij}} = c_{ij} \quad | \quad mn \text{ equations}$$

$$\left.\begin{aligned} w_{s_{ij}} &\geq 0 \quad | \quad mn \text{ inequalities} \\ u_i \geq 0\,,\; v_j &\geq 0 \quad | \quad m + n \text{ inequalities} \end{aligned}\right\} (7\bar{D}) \qquad (16)$$

Let $(u_1^o, \ldots, u_m^o, v_1^o, \ldots, v_n^o)'$ be the solution to Eqs 15. Since Eq 16 must hold for optimality we must have $w_{s_{ij}}^o \geq 0\,, u_i^o \geq 0\,, v_j^o \geq 0$ [$(mn + m + n)$ inequalities] where

$$w_{s_{ij}}^o = d_{ij}u_i^o + v_j^o - c_{ij} \qquad (17)$$

If $w_{s_{ij}}^o \geq 0\,, u_i^o \geq 0\,, v_j^o \geq 0$ for all i,j then X^o is optimal and the algorithm is terminated. If, however, $w_{s_{ij}}^o < 0$ for some i,j or $u_i^o < 0$ or $v_j^o < 0$ for some i or j, an improvement in the solution X^o can be made.

4. Nonoptimal Solutions; Finding the Greatest Violation of the Dual Space Constraints; Identifying the Associated Vector for Entry into the Primal Solution Space Basis

The greatest violation, V, of the dual space constraints (Conditions $7\bar{D}$) is simply

$$V = \min \left\{ \min_{i,j} \{ w_{s_{ij}}^o \mid w_{s_{ij}}^o < 0 \} \;;\; \min_i \{ u_i^o \mid u_i^o < 0 \} \;;\; \min_j \{ v_j^o \mid v_j^o < 0 \} \right\} \qquad (18)$$

Depending on which of the above three bracketed minimums is largest in magnitude, the corresponding vector chosen to enter the new basis is one of the three types of vectors $d_{ij}\vec{e}_i + \vec{e}_{m+j}$, $\vec{e}_i$, or $\vec{e}_{m+j}$.

5. Finding the Representation of the New Basis Vector in Terms of the Current Basis Vectors

Consider the three cases (a) $V = w_{s_{\bar{i}\bar{j}}}^o$, (b) $V = u_{\bar{i}}^o$, (c) $V = v_{\bar{j}}^o$ that can result from Eq 18.† The vector equation to be solved for a singly indexed system is

$$\vec{A}_k = By_k \quad \text{or} \quad y_k = B^{-1}\vec{A}_k \qquad (19)$$

†The bar denotes the minimizing index or indexes in Eq 18.

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where y_k is the vector of coordinates of $\vec{A}_k$ relative to the basis of column vectors of B.

Corresponding to cases a, b, or c the following vector equation is solved for $y_{ij}^{\bar{i}\bar{j}}$, the $m + n$ components of the entering basis vector.

$$\left.\begin{array}{ll}\text{(a)} & d_{\bar{i}\bar{j}}\vec{e}_{\bar{i}} + \vec{e}_{m+\bar{j}} \\ \text{(b)} & e_{\bar{i}} \\ \text{(c)} & \vec{e}_{m+\bar{j}}\end{array}\right\} = \sum_{\substack{i,j \\ x_{ij} > 0}} y_{ij}^{\bar{i}\bar{j}}(d_{ij}\vec{e}_i + \vec{e}_{m+j}) + \sum_{\substack{i \\ x_{in'} > 0}} y_{in'}^{\bar{i}\bar{j}}\vec{e}_i + \sum_{\substack{j \\ x_{m'j} > 0}} y_{m'j}^{\bar{i}\bar{j}}\vec{e}_{m+j} \qquad (20)$$

Equation 20 leads to the following three sets of scalar equations in $y_{ij}^{\bar{i}\bar{j}}$, $y_{ij}^{\bar{i}n'}$, or $y_{ij}^{m'\bar{j}}$:

For Eq 20a

$$\begin{array}{lll} \displaystyle\sum_{\substack{j \\ x_{ij} > 0}} y_{ij}^{\bar{i}\bar{j}} d_{ij} + y_{in'}^{\bar{i}\bar{j}} = d_{\bar{i}\bar{j}} & \text{if } i = \bar{i} & \\ & & (i = 1, \ldots, m) \\ \displaystyle\sum_{\substack{j \\ x_{ij} > 0}} y_{ij}^{\bar{i}\bar{j}} d_{ij} = 0 & \text{if } i \neq \bar{i} & \\ \displaystyle\sum_{\substack{i \\ x_{ij} > 0}} y_{ij}^{\bar{i}\bar{j}} + y_{m'j}^{\bar{i}\bar{j}} = 1 & \text{if } j = \bar{j} & \\ & & (j = 1, \ldots, n) \\ \displaystyle\sum_{\substack{i \\ x_{ij} > 0}} y_{ij}^{\bar{i}\bar{j}} = 0 & \text{if } j \neq \bar{j} & \end{array} \qquad (21)$$

For Eq 20b

$$\begin{array}{ll} \displaystyle\sum_{\substack{j \\ x_{ij} > 0}} y_{ij}^{\bar{i}n'} d_{ij} = \begin{cases} 1 & \text{if } i = \bar{i} \\ 0 & \text{if } i \neq \bar{i} \end{cases} & (i = 1, \ldots, m) \\ \displaystyle\sum_{\substack{i \\ x_{ij} > 0}} y_{ij}^{\bar{i}n'} = 0 & (j = 1, \ldots, n) \end{array} \qquad (22)$$

For Eq 20c

$$\begin{array}{ll} \displaystyle\sum_{\substack{j \\ x_{ij} > 0}} y_{ij}^{m'\bar{j}} d_{ij} = 0 & (i = 1, \ldots, m) \\ \displaystyle\sum_{\substack{i \\ x_{ij} > 0}} y_{ij}^{m'\bar{j}} = \begin{cases} 1 & \text{if } j = \bar{j} \\ 0 & \text{if } j \neq \bar{j} \end{cases} & (j = 1, \ldots, n) \end{array} \qquad (23)$$

6. Selecting the Vector To Leave the Current Basis

Once the vector that enters the new basis has been found, the associated positive components of the new primal solution X^o_{new} must also satisfy Conditions 7P for primal feasibility. Hence we have

$$BX^o = B_{new}X^o_{new} = b \qquad X^o_{new} \geq 0 \tag{24}$$

or

$$BX^o - \theta \left\{\begin{matrix} (a) & d_{\bar i \bar j}\vec{e}_{\bar i} + \vec{e}_{m+\bar j} \\ (b) & \vec{e}_{\bar i} \\ (c) & \vec{e}_{m+\bar j} \end{matrix}\right\} + \theta \left\{\begin{matrix} (a) & d_{\bar i \bar j}\vec{e}_{\bar i} + \vec{e}_{m+\bar j} \\ (b) & \vec{e}_{\bar i} \\ (c) & \vec{e}_{m+\bar j} \end{matrix}\right\} = b \tag{25}$$

$$= BX^o - \theta \left\{\begin{matrix} (a) & By^{\bar i \bar j} \\ (b) & By^{\bar i n'} \\ (c) & By^{m' \bar j} \end{matrix}\right\} + \theta \left\{\begin{matrix} (a) & d_{\bar i \bar j}\vec{e}_{\bar i} + \vec{e}_{m+\bar j} \\ (b) & \vec{e}_{\bar i} \\ (c) & \vec{e}_{m+\bar j} \end{matrix}\right\} = b \tag{26}$$

$$= B \left\{ X^o - \begin{matrix} (a) & \theta y^{\bar i \bar j} \\ (b) & \theta y^{\bar i n'} \\ (c) & \theta y^{m' \bar j} \end{matrix}\right\} + \theta \left\{\begin{matrix} (a) & d_{\bar i \bar j}\vec{e}_{\bar i} + \vec{e}_{m+\bar j} \\ (b) & \vec{e}_{\bar i} \\ (c) & \vec{e}_{m+\bar j} \end{matrix}\right\} = b \tag{27}$$

Since $X^o_{new} > 0$ we have in particular (a) $x^o_{\bar i \bar j} = \theta > 0$, or (b) $x^o_{\bar i n'} = \theta > 0$, or (c) $x^o_{m' \bar j} = \theta > 0$ corresponding to the new basis vector a, b, or c. The remaining $m + n - 1$ column vectors of B_{new} are determined by eliminating that column vector of B whose new associated primal solution component $x^o_{ij_{new}}$ vanishes.

This elimination is accomplished as follows. Writing the expressions in the left braces of Eq 27 in component form, we have

$$x^o_{ij_{new}} = \left\{ x^o_{ij} - \begin{matrix} (a) & \theta y^{\bar i \bar j}_{ij} \\ (b) & \theta y^{\bar i n}_{ij} \\ (c) & \theta y^{m' \bar j}_{ij} \end{matrix}\right\} \quad \text{for all } (i,j) \ni x_{ij} > 0 \tag{28a}$$

$$x^o_{in'_{new}} = x^o_{in'} - \theta y^{\bar i \bar j}_{in'} \quad \text{for all } (i,n') \ni x_{in'} > 0 \tag{28b}$$

$$x^o_{m'j_{new}} = x^o_{m'j} - \theta y^{\bar i \bar j}_{m'j} \quad \text{for all } (m',j) \ni x_{m'j} > 0 \tag{28c}$$

Since we want $x^o_{ij_{new}}$, or $x_{in'_{new}}$, or $x_{m'j_{new}}$ to vanish we select positive θ from Eq 29

$$\theta = \hat{\theta} = \min_{ij}\left\{ \frac{x^{o}_{ij}}{y^{\bar{i}\bar{j}}_{ij}} \;\middle|\; y^{\bar{i}\bar{j}}_{ij} > 0 \right\} \quad \begin{matrix} (i = 1, \ldots, m') \\ (j = 1, \ldots, n') \end{matrix} \tag{29}$$

If the minimizing indexes in Eq 29 are $(i,j) = (p,q)$ then $x^{o}_{pq_{new}} = 0$ and $d_{pq}\vec{e}_p + \vec{e}_{m+q}$ leaves the basis if $p \neq m'$ or $q \neq n'$, $\vec{e}_p$ leaves the basis if $q = n'$, and $\vec{e}_{m+q}$ leaves the basis if $p = m'$.

7. Expressing the Solution in Terms of the New Basis

The new solution X^{o}_{new} has components expressed by Eqs 28a to 28c with $\theta = \hat{\theta}$. In particular, as mentioned before, $x^{o}_{pq_{new}} = 0$ and $x^{o}_{\bar{i}\bar{j}} = \hat{\theta}$ for the primal variables associated with the leaving and entering vectors respectively.

8. Return to Step 2

Self-explanatory.

C. EFFECT OF NEW SOLUTIONS ON VALUE OF OBJECTIVE FUNCTION

There is associated with any violation of Conditions $7\bar{D}$ a new solution (X^{o}_{new}) to Conditions 7P that improves the value of the objective function $\langle c^{o}, X^{o}\rangle$ and at the same time eliminates the specific violation of $7\bar{D}$.

Recall from step 6 of the algorithm Conditions 7P are preserved when a new vector $\vec{A}_k$ enters the basis, thus

$$\begin{aligned} BX^{o} - \theta A_k + \theta\vec{A}_k &= b \\ BX^{o} - \theta By_k + \theta\vec{A}_k &= b \qquad \vec{A}_k = By_k \\ B(X^{o} - \theta y_k) + \theta\vec{A}_k &= b \end{aligned} \tag{30}$$

For the corresponding expression to the objective function value we have

$$\begin{aligned} &\langle c^{o},(X^{o} - \theta y_k)\rangle + \theta c_k \\ \text{or}\quad &\langle c^{o},X^{o}\rangle + \theta(c_k - \langle c^{o},y_k\rangle) \quad \text{(new objective function value)} \end{aligned} \tag{31}$$

The term $(c_k - \langle c^{o},y_k\rangle)$ corresponds to $(c_k - z_k)$ in general simplex notation and in the notation of this paper to (a) $-w_{s_k}$ for $1 \le k \le mn$ when $\vec{A}_k = d_{ij}\vec{e}_i + \vec{e}_{m+j}$, $k = n(i-1) + j$, or (b) $-u_i$ when $\vec{A}_k = \vec{e}_i$, $mn < k \le mn + m$, or (c) $-v_j$ when $\vec{A}_k = \vec{e}_{m+j}$

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$mn + m < k \le mn + m + n$. Thus for positive θ and any violation of Conditions 7$\overline{D}$, i.e., $w_{s_k} < 0$, $u_i < 0$, or $v_j < 0$, there is an associated improvement in the objective function value of magnitude (a) $-\theta w_{s_k}$, (b) $-\theta u_i$, or (c) $-\theta v_j$ when the vector (a) $\vec{A}_k$, (b) $\vec{e}_i$, or (c) $\vec{e}_{m+j}$ enters the new basis. Condition 7$\overline{S}$ guarantees that the violation will be eliminated for the next iteration.

Throughout the algorithm values for z_k ($z_k = \langle c^o, y_k \rangle$) are not computed using the y_k representation of $\vec{A}_k$ (i.e., the representation relative to basis vectors B), but from the relation $z_k = \langle w^o, \vec{A}_k \rangle$ which makes for greater efficiency in computation.

8. COMPUTATIONAL EXPERIENCE

The algorithm briefly called MATMAX was originally used to solve the linear subproblems described in Ref 1 with $m = 3$, $n = 4$. During the process of convergence to a single larger nonlinear programming problem solution to which the linear programming Prob 3 is only a constraint, it became necessary to solve the linear problems in the order of ten thousand times. The need for an algorithm faster than the standard simplex algorithm thus arose.

TABLE 1

Solution Times for MATMAX and Standard Simplex Algorithms[a,3]

Number of constraints m	Number of constraints n	MATMAX, sec	Simplex,[7] sec	Simplex,[b] sec
5	4	0.16	0.60	0.36
10	12	5.56	31.51	15.84
18	24	24.88	297.10	114.98

[a]Solution times are based on single precision operations in FORTRAN IV using the IBM 7044 computer.
[b]See App C.

The MATMAX algorithm has been compared for solution time with the simplex algorithm[7] and an even faster simplex algorithm given in App C. The A matrix (with identity) requires 19,988 = 42 × 474 storage locations for the $m = 18$, $n = 24$ simplex algorithm, thus limiting the size of "incore" comparison of the algorithms. Solution times for MATMAX and standard simplex algorithms are shown in Table 1.

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The success of the algorithm currently depends on being able to solve the $m + n$ linear equations Eqs 15 in u_i and v_j sequentially, i.e., on solving for the nonzero u_i and v_j in terms of zero valued u_i and/or v_j.

If, however, it is not possible to solve the system of Eqs 15 sequentially during some iteration of the algorithm,an attempt is made to bypass the difficulty. The algorithm then attempts to proceed to the optimum by avoiding the particular vertex for which Eqs 15 could not be solved. Currently the algorithm returns to phase 1 (the matrix maximum phase),this time assigning one less x_{ij} having positive cost c_{ij} and one more positive slack variable (partial matrix maximum) than was assigned the prior solution of phase 1. Beginning with this solution in phase 2 (simplex) a new sequence of vertexes is generated for which the problem of nonsequential solvability of Eqs 15 is frequently avoided.

The above process has worked successfully on most problems of moderate size but has failed on one with $m = 30$, $n = 32$. In some situations more than one return to phase 1 may be required in order to find a sequence of vertexes for which Eqs 15 may be solved.

It is, of course, possible to solve Eqs 15 when sequential methods fail. However, the logic of loop-tracing techniques required in such situations is complex and is not currently employed. Alternative methods that use the sequential solvability of Eqs 15 as a secondary criterion for selecting the next neighboring vertex are under investigation.

APPENDIXES

Appendix A

COMPUTER PROGRAM FOR THE ALGORITHM

INTRODUCTION

A listing of the computer program for the algorithm follows. The algorithm has been used successfully on problems of moderate size (see Sec 8, "Computational Experience"). The comment cards identify appropriate subsections of the program as described in Sec 6, "Details of the Algorithm."

The success of the algorithm currently depends on being able to solve the $m + n$ linear equations Eqs 15 in u_i and v_j sequentially, i.e., on solving for the nonzero u_i and v_j in terms of zero valued u_i and/or v_j, as discussed in "Computational Experience."

Several working arrays are used for calculations; namely, the AE, ABASIS, YY, ABAR, BBAR, U, V, ISUM, and JSUM arrays.

The value of the objective function is printed out every kth iteration by setting IWRITE = k on the first input card. In addition, a detailed printout will be given every kth iteration by setting ITAB = 1 on the same card.

The computer program of App C accepts exactly the same input cards as the following program, with the exception as stated in App C.

The subroutine TODAY called twice in the program is used for timing purpose only. The general user should not call this subroutine.

PROGRAM

```
                                                FORTRAN SOURCE LIST
ISN         SOURCE STATEMENT

  0  $IBFTC MATMAX
  1         DIMENSION CE(35,35),DE(36,36),AE(36,36),ABASIS(36,36),YY(36,36)
  2         DIMENSION ABAR(36),BBAR(36),ISUM(36),JSUM(36)
  3         DIMENSION AA(36),BB(36)
  4         DIMENSION XX(35,35),U(35),V(35)
  5  19001  FORMAT(1H1)
  6  19002  FORMAT(1H )
  7  19003  FORMAT(4I10)
 10  19004  FORMAT(5F12.6)
 11  19005  FORMAT(15X,6E16.6)
 12  19006  FORMAT(15X,4HMS= ,I2,4X,4HNS= ,I2,15X,30HFREQ. OF OBJ. FUNCT. PRIN
           1TOUT ,I2,5X,29HDETAILED PRINTOUT IF ITAB = 1,5X,5HITAB=,I2//)
 13  19007  FORMAT(30X,46HINPUT CONSTANTS FOR OPTIMAL ALLOCATION PROBLEM//)
 14  19008  FORMAT(15X,24HINPUT VALUES FOR CE(I,J)//)
 15  19009  FORMAT(15X,24HINPUT VALUES FOR DE(I,J)//)
 16  19010  FORMAT(15X,45HINPUT VALUES FOR AA(I) OF THE ROW CONSTRAINTS//)
 17  19011  FORMAT(15X,48HINPUT VALUES FOR BB(J) OF THE COLUMN CONSTRAINTS//)
 20  19012  FORMAT( 52HDEGENERACY OCCURS FOR ABAR(I) BBAR(J) TRY NEW DELTA1)
 21  19013  FORMAT( 7HIMAX = ,I3,7HJMAX = ,I3)
 22  19014  FORMAT( 13HABAR(IMAX) = ,E14.8,13HBBAR(JMAX) = ,E14.8)
 23  19015  FORMAT(//37HTOTAL EXECUTION TIME FOR ALGORITHM = ,F12.7.1X,4HSEC.)
 24  19016  FORMAT(8I10)
 25  19017  FORMAT(40X,23HDETAILS OF THE SOLUTION//)
 26  19018  FORMAT(//44HNUMBER OF ITERATIONS AFTER INITIAL SOLUTION ,I5)
 27  19019  FORMAT(//34HVALUE OF PRIMAL OBJECTIVE FUNCTION,E20.6)
 30  19020  FORMAT(//32HVALUE OF DUAL OBJECTIVE FUNCTION,E20.6)
 31  19021  FORMAT(10X,4HROW ,I2,5X,7HCOLUMN ,I2,5X,11HALLOCATION ,E12.6,5X,23
           1HRETURN FROM ALLOCATION ,E12.6)
 32  19022  FORMAT(//74HVALUES OF  THE DUAL SPACE VARIABLES (LAGRANGE MULTIPLI
           1ERS , SHADOW PRICES))
 33  19023  FORMAT(10X,4HROW ,I2,5X,6HU(I)= ,E12.6)
 34  19024  FORMAT(7X,7HCOLUMN ,I2,5X,6HV(J)= ,E12.6)
 35  19025  FORMAT(//37HMAXIMUM VIOLATION OF DUAL CONSTRAINTS,E15.6)
 36  19026  FORMAT(//10X,28HSTARTING NEW SEQUENCE NUMBER.I4)
 37  19027  FORMAT(6X,4H ROW,I3)
 40  19028  FORMAT(10X,41HTOTAL ITERATIONS IN LAST VERTICE SEQUENCE,I3)
 41  19029  FORMAT(10X,47HNUMBER OF POS. XX(I,J) TO BE ASSIGNED BY MATMAX,I3)
 42  19030  FORMAT(5X,10HITERATION ,I4,8X,28HPRIOR VALUE OF OBJ FUNCTION ,E12.
           16,8X,31HMAX. VIOLATION OF DUAL CONSTR. ,E12.6/)
 43  19031  FORMAT(//32HMATRIX MAXIMUM ITERATION NUMBER ,I4)
 44  19032  FORMAT(5X,19HALLOCATION SELECTED,4X,4HROW ,I4,4X,5HCOL. ,I4,4X,9HX
           1X(I,J)= ,F12.6)
 45  19033  FORMAT(5X15HENTERING VECTOR,2I5,8X,14HLEAVING VECTOR,2I5,8X,20HXX(
           1IENTER,JENTER) = ,F12.6//)
 46  19034  FORMAT(5X,30HVALUE OF OBJECTIVE FUNCTION = ,E12.6)
 47  19035  FORMAT(///35X,30HDETAILED INTERMEDIATE PRINTOUT/)
 50  19036  FORMAT(/10X,30HCURRENT SOLUTION ARRAY XX(I,J)/)
 51  19037  FORMAT(/15X,16HUNUSED RESOURCES,5X,26HCOLUMNS 1 THRU NS IN ORDER/)
 52  19038  FORMAT(/15X,17HUNUSED CAPACITIES,5X,23HROWS 1 THRU MS IN ORDER/)
 53  19039  FORMAT(/15X,50H  PRIOR VALUES OF THE DUAL VARIABLES U(I) IN ORDER)
 54  19040  FORMAT(/15X,50H  PRIOR VALUES OF THE DUAL VARIABLES V(J) IN ORDER)
 55         WRITE(6,19001)
 56         WRITE(6,19007)
 57         READ(5,19003)MS,NS,IWRITE,ITAB
 64         WRITE(6,19006)MS,NS,IWRITE,ITAB
```


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```
ISN        SOURCE STATEMENT

 65          WRITE(6,19002)
 66          WRITE(6,19008)
 67          DO 19050 I=1,MS
 70          READ(5,19004)(CE(I,J),J=1,NS)
 75          WRITE(6,19027)I
 76  19050   WRITE(6,19005)(CE(I,J),J=1,NS)
104          WRITE(6,19002)
105          WRITE(6,19009)
106          DO 19060 I=1,MS
107          READ(5,19004)(DE(I,J),J=1,NS)
114          WRITE(6,19027)I
115  19060   WRITE(6,19005)(DE(I,J),J=1,NS)
123          WRITE(6,19002)
124          WRITE(6,19010)
125          READ(5,19004)(AA(I),I=1,MS)
132          WRITE(6,19005)(AA(I),I=1,MS)
137          WRITE(6,19002)
140          WRITE(6,19011)
141          READ(5,19004)(BB(J),J=1,NS)
146          WRITE(6,19005)(BB(J),J=1,NS)
153          WRITE(6,19001)
154           CALL TODAY(0,ITIME,IDAT)
      C---GENERATION OF BASIC FEASIBLE SOLUTION USING MATRIX MAXIMUM-19100-199
155          MTOTAL = MS * NS
156          MBIG   = MS + NS
157          DELTA1 = .1E-5
160          DELTA2 = .1E-4
161          NSS = NS + 1
162          MSS = MS + 1
163          MOUT = MBIG
164  19100   II = 0
165          TEMP2 = 0.
166          MMAT = 0
167          DO 19105 I=1,MS
170  19105   ABAR(I)=AA(I)
172          DO 19110 J=1,NS
173  19110   BBAR(J)=BB(J)
175          DO 19199 L=1,MOUT
176          AMAX = 0.
177          DO 19140 I=1,MS
200          IF(ABAR(I)-.1E-6)19140,19140,19120
201  19120   DO 19135 J=1,NS
202          IF(BBAR(J) - .1E-6)19135,19135,19125
203  19125   IF(CE(I,J) - AMAX)19135,19135,19130
204  19130   AMAX = CE(I,J)
205          IMAX = I
206          JMAX = J
207  19135   CONTINUE
211  19140   CONTINUE
213          IF(AMAX)19200,19200,19150
214  19150   ABARTP =  ABAR(IMAX)/ DE(IMAX,JMAX)
215          AE(IMAX,JMAX) = CE(IMAX,JMAX)
216          BBARTP = BBAR(JMAX)
217          IF(ABARTP - BBARTP)19160,19170,19180
      C---BRANCH 19170 IS FOR  DEGENERACY ---
```

```
ISN        SOURCE STATEMENT

     19160 XX(IMAX,JMAX)=ABARTP
           ABAR(IMAX)= 0.
           BBAR(JMAX)= BBARTP - ABARTP
           IF(ITAB - 1)19199,19165,19199
     19165 TEMP2 = TEMP2 + CE(IMAX,JMAX) * XX(IMAX,JMAX)
           MATMA = MMAT + 1
           WRITE(6,19031)MATMA
           WRITE(6,19032)IMAX,JMAX,XX(IMAX,JMAX)
           WRITE(6,19034)TEMP2
           GO TO 19199
     19170 ABAR(IMAX)= ABAR(IMAX) + DELTA1
           IDEGEN = IDEGEN + 1
           IF(IDEGEN - MTOTAL)19150,19150,19175
     19175 WRITE(6,19012)
           WRITE(6,19013)IMAX,JMAX
           WRITE(6,19014)ABAR(IMAX),BBAR(JMAX)
           IDEGEN = 0
           DELTA1 = 10. * DELTA1
           GO TO 19170
     19180 XX(IMAX,JMAX)=BBARTP
           BBAR(JMAX) =0.
           ABAR(IMAX) = ABAR(IMAX) - DE(IMAX,JMAX)* BBARTP
           IF(ITAB - 1)19199,19185,19199
     19185 TEMP2 = TEMP2 + CE(IMAX,JMAX) * XX(IMAX,JMAX)
           MATMA = MMAT + 1
           WRITE(6,19031)MATMA
           WRITE(6,19032)IMAX,JMAX,XX(IMAX,JMAX)
           WRITE(6,19034)TEMP2
     19199 MMAT = MMAT + 1
     C------SOLVE FOR THE DUAL SPACE VARIABLES U(I) AND V(J) 19200-19299
     19200 IDUAL = 0
           DO 19206 I=1,MS
           IF(ABAR(I) - .1E-6)19202,19202,19204
     19202 U(I)= 1.E+35
           GO TO 19206
     19204 U(I)= 0.
           IF(AA(I) - ABAR(I))19206,19206,19205
     19205 IDUAL = IDUAL + 1
     19206 CONTINUE
           DO 19212 J=1,NS
           IF(BBAR(J) - .1E-6)19208,19208,19210
     19208 V(J)=1.E+35
           GO TO 19212
     19210 V(J) = 0.
           IF(BB(J) - BBAR(J))19212,19212,19211
     19211 IDUAL = IDUAL + 1
     19212 CONTINUE
           IF(IDUAL - 1)19215,19219,19219
     19215 MOUT = MMAT - 1
           IF(MOUT - 2)19411,19216,19216
     19216 DO 19218 I=1,MSS
           DO 19217 J=1,NSS
           AE(I,J) = 0.
     19217 CONTINUE
     19218 CONTINUE
```


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```
ISN        SOURCE STATEMENT

                 IVERT = IVERT + 1
                 WRITE(6,19026)IVERT
                 WRITE(6,19028)I1
                 WRITE(6,19029)MOUT
                 IPRINT = 0
                 GO TO 19100
           19219 IFINAL = 0
                 MDUAL = 0
           19220 IF(IFINAL - MS)19222,19300,19300
           19222 MDUAL = MDUAL + 1
                 IF(MDUAL - MBIG)19225,19225,19215
           19225 IFINAL = 0
                 DO 19290 I=1,MS
                 IF( U(I) - 1.E+35)19260,19230,19260
           19230 DO 19250 J=1,NS
                 IF( V(J) - 1.E+35)19235,19250,19235
           19235 IF(AE(I,J))19250,19250,19240
           19240 U(I) = (CE(I,J) - V(J))/ DE(I,J)
                 GO TO 19290
           19250 CONTINUE
                 GO TO 19290
           19260 IFINAL = IFINAL + 1
                 DO 19280 JJ =1,NS
                 IF(V(JJ) - 1.E+35)19280,19265,19280
           19265 IF(AE(I,JJ))19280,19280,19270
           19270 V(JJ) = CE(I,JJ) - DE(I,JJ) * U(I)
           19280 CONTINUE
           19290 CONTINUE
                 GO TO 19220
           C---OPTIMALITY  TEST  19300-19399-------
           19300 DIFMIN = 0.
                 DO 19302 I=1,MS
                 IF(U(I) - DIFMIN)19301,19302,19302
           19301 DIFMIN = U(I)
                 IENTER = I
                 JENTER = NSS
           19302 CONTINUE
                 DO 19304 J =1,NS
                 IF(V(J) - DIFMIN)19303,19304,19304
           19303 DIFMIN = V(J)
                 JENTER = J
                 IENTER = MSS
           19304 CONTINUE
                 DO 19321 I=1,MS
                 DO 19320 J=1,NS
                 IF(AE(I,J))19320,19305,19320
           19305 DIF = DE(I,J) * U(I) + V(J) - CE(I,J)
                 IF(DIF)19310,19320,19320
           19310 IF(DIF - DIFMIN)19315,19320,19320
           19315 DIFMIN = DIF
                 IENTER = I
                 JENTER = J
           19320 CONTINUE
           19321 CONTINUE
                 IF(DIFMIN + DELTA2)19400,19325,19325
```

```
ISN       SOURCE STATEMENT

    C---EXIT 19325 IS FOR OPTIMAL SOLUTIONS----ALL DIF ARE NON NEGATIVE
      19325 CONTINUE
             CALL  TODAY(1,ITIME,IDAT)
             TIME  = FLOAT(ITIME)/60.
            WRITE(6,19015)TIME
            WRITE(6,19018)I1
            WRITE(6,19025)DIFMIN
            WRITE(6,19001)
            WRITE(6,19017)
            PRIMAL  =  0.
            DUAL    =  0.
            DO 19340 I=1,MS
            DO 19335 J=1,NS
            IF(AE(I,J))19335,19335,19330
      19330 TEMP  = XX(I,J)*CE(I,J)
            PRIMAL  = PRIMAL + TEMP
            WRITE(6,19021)I,J,XX(I,J),TEMP
      19335 CONTINUE
      19340 CONTINUE
            WRITE(6,19019)PRIMAL
            WRITE(6,19022)
            DO 19355  I=1,MS
            WRITE(6,19023)I,U(I)
            DUAL  = DUAL+ AA(I) * U(I)
      19355 CONTINUE
            DO 19360  J=1,NS
            WRITE(6,19024)J,V(J)
            DUAL= DUAL + BB(J) * V(J)
      19360 CONTINUE
            WRITE(6,19020)DUAL
            CALL EXIT
    C---REPRESENTATION OF ENTERING VECTOR BY CURRENT BASIS 19400-19499---
      19400 ABAR(MSS)= 0.
            BBAR(NSS)= 0.
            NSPACE  = 0
            DO 19401 I=1,MSS
            ISUM(I)= 0
            AE(I,NSS)=ABAR(I)
      19401 DE(I,NSS)=0.
            DO 19402 J=1,NSS
            JSUM(J)= 0
            AE(MSS,J)  = BBAR(J)
      19402 DE(MSS,J)  = 0.
            DO 19405 I=1,MSS
            DO 19404 J=1,NSS
            ABASIS(I,J)  = AE(I,J)
            YY(I,J)  = 0.
            IF(AE(I,J))19404,19404,19403
      19403 ISUM(I)  = ISUM(I)+ 1
            JSUM(J)  = JSUM(J)+ 1
            NSPACE  = NSPACE + 1
      19404 CONTINUE
      19405 CONTINUE
            I1=I1+1
            IF(IENTER - MSS)19407,19406,19407
```

```
ISN        SOURCE STATEMENT

505  19406 DE(IENTER,NSS)=0.
506        DE(MSS,JENTER) = 1.
507        GO TO 19410
510  19407 IF(JENTER - NSS)19409,19408,19409
511  19408 DE(IENTER,NSS)=1.
512        DE(MSS,JENTER) = 0.
513        GO TO 19410
514  19409 DE(MSS,JENTER) = 1.
515        DE(IENTER,NSS) = DE(IENTER,JENTER)
516  19410 IF(II - MTOTAL)19414,19414,19411
517  19411 WRITE(6,19005)DIFMIN
520        WRITE(6,19003)II
521        WRITE(6,19016)(ISUM(I),I=1,MS)
526        WRITE(6,19016)(JSUM(J),J=1,NS)
533        DO 19412 I=1,MSS
534  19412 WRITE(6,19004)(AE(I,J),J=1,NSS)
542        WRITE(6,19016)IENTER,JENTER
543        DO 19413 I = 1,MSS
544  19413 WRITE(6,19004)(ABASIS(I,J),J=1,NSS)
552        GO TO 19325
553  19414 CONTINUE
554        NTOTAL = 0
555        IREPRE = 0
556  19415 IF(NTOTAL - MBIG)19417,19495,19495
557  19417 IREPRE = IREPRE + 1
560        IF(IREPRE - MBIG)19420,19420,19411
561  19420 DO 19455 I=1,MS
562        IF(ISUM(I) - 1)19455,19425,19455
563  19425 DO 19445 J=1,NSS
564        IF(ABASIS(I,J))19445,19445,19430
565  19430 IF(J - NSS)19435,19440,19435
566  19435 YY(I,J)= DE(I,NSS)/DE(I,J)
567        DE(I,NSS)= YY(I,J)
570        DE(MSS,J)= DE(MSS,J) - YY(I,J)
571        GO TO 19450
572  19440 YY(I,J)= DE(I,NSS)
573        GO TO 19450
574  19445 CONTINUE
576  19450 ISUM(I)= ISUM(I)- 1
577        JSUM(J)= JSUM(J)- 1
600        NTOTAL = NTOTAL +1
601        ABASIS(I,J)= 0.
602  19455 CONTINUE
604        IF(NTOTAL - MBIG)19460,19495,19495
605  19460 DO 19490 J=1,NS
606        IF(JSUM(J) - 1)19490,19470,19490
607  19470 DO 19480 I=1,MSS
610        IF(ABASIS(I,J))19480,19480,19475
611  19475 YY(I,J)= DE(MSS,J)
612        DE(I,NSS) = DE(I,NSS)-  DE(I,J)* YY(I,J)
613        GO TO 19485
614  19480 CONTINUE
615  19485 ISUM(I)= ISUM(I)-1
617        JSUM(J)= JSUM(J)-1
620        ABASIS(I,J) = 0.
```


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```
ISN        SOURCE STATEMENT

621              NTOTAL = NTOTAL + 1
622 19490  CONTINUE
624              GO TO 19415
625 19495  CONTINUE
      C---VECTOR TO LEAVE BASIS IS NOW DETERMINED----19500 - 19599
626 19500  THETA   = 1.E+35
627              DO 19525 I=1,MSS
630              DO 19520  J=1,NSS
631              IF(AE(I,J))19520,19520,19504
632 19504  IF(YY(I,J))19520,19520,19506
633 19506  IF(I - MSS)19510,19508,19510
634 19508  ALTEMP = BBAR(J)/YY(I,J)
635              GO TO 19516
636 19510  IF(J - NSS)19514,19512,19514
637 19512  ALTEMP = ABAR(I)/YY(I,J)
640              GO TO 19516
641 19514  ALTEMP = XX(I,J)/YY(I,J)
642 19516  IF(ALTEMP -   THETA)19518,19520,19520
643 19518  THETA   =   ALTEMP
644              ILEAVE  = I
645              JLEAVE  = J
646 19520  CONTINUE
650 19525  CONTINUE
      C---TRANSFORM OLD SOLUTION IN TERMS OF CURRENT BASIS 19600-19699 ----
      C---NEW SOLUTION IS RETURNED TO 19200 -----
652 19600  DO 1962O  I=1,MS
653              DO 19615 J=1,NS
654              IF(AE(I,J))19615,19615,19605
655 19605  IF(YY(I,J))19610,19615,19610
656 19610  XX(I,J)  =  XX(I,J)  -  THETA*YY(I,J)
657 19615  CONTINUE
661 19620  CONTINUE
663              DO 19630  I=1,MS
664              IF(YY(I,NSS))19625,19630,19625
665 19625  ABAR(I)  =  ABAR(I)  -  THETA   * YY(I,NSS)
666 19630  CONTINUE
670              DO 19640  J=1,NS
671              IF(YY(MSS,J))19635,19640,19635
672 19635  BBAR(J)  =  BBAR(J)  -  THETA   * YY(MSS,J)
673 19640  CONTINUE
675              AE(ILEAVE,JLEAVE)= 0.
676              IF(ILEAVE  -  MSS)19642,19641,19642
677 19641  BBAR(JLEAVE)  =  0.
700              GO TO 19644
701 19642  IF(JLEAVE  -  NSS)19644,19643,19644
702 19643  ABAR(ILEAVE)  =  0.
703 19644  IF(IENTER  -  MSS)19650,19645,19650
704 19645  BBAR(JENTER)  =  THETA
705              AE(MSS,JENTER)  =  THETA
706              GO TO 19700
707 19650  IF(JENTER  -  NSS)19660,19655,19660
710 19655  ABAR(IENTER)  =  THETA
711              AE(IENTER,NSS)  =  THETA
712              GO TO 19700
713 19660  XX(IENTER,JENTER)  =  THETA
```


RAC

```
ISN        SOURCE STATEMENT

714           AE(IENTER,JENTER)= CE(IENTER,JENTER)
715           IF(ILEAVE - MSS)19675,19700,19675
716  19675    IF(JLEAVE - NSS)19680,19700,19700
717  19680    XX(ILEAVE,JLEAVE) = 0.
     C-----FREQUENCY OF OBJ. FUNCTION PRINTOUT CONTROLED BY SETTING IWRITE-------
720  19700    IPRINT = IPRINT + 1
721           IF(IPRINT - IWRITE)19200,19710,19710
722  19710    IPRINT = 0
723           DUAL = 0.
724           DO 19750 I=1,MS
725  19750    DUAL = DUAL+ AA(I) * U(I)
727           DO 19760 J=1,NS
730  19760    DUAL= DUAL + BB(J) * V(J)
732           WRITE(6,19039)II,DUAL,DIFMIN
     C-----DETAILED PRINTOUT CONTROLED BY SETTING ITAB = 1 IN INPUT-------------
733  19800    IF(ITAB - 1)19200,19810,19200
734  19810    WRITE(6,19001)
735           WRITE(6,19035)
736           WRITE(6,19039)
737           WRITE(6,19005)(U(I),I=1,MS)
744           WRITE(6,19040)
745           WRITE(6,19005)(V(J),J=1,NS)
752           WRITE(6,19033)IENTER,JENTER,ILEAVE,JLEAVE,THETA
753           WRITE(6,19036)
754           DO 19820 I=1,MS
755           WRITE(6,19027)I
756  19820    WRITE(6,19005)(XX(I,J),J=1,NS)
764           WRITE(6,19037)
765           WRITE(6,19005)(BBAR(J),J=1,NS)
772           WRITE(6,19038)
773           WRITE(6,19005)(ABAR(I),I=1,MS)
1000          GO TO 19200
1001          END
```

Appendix B

SAMPLE PROBLEM

EXAMPLE

A short example is given here to illustrate (a) the type of problem to which the algorithm can be applied and (b) the details of the solution process. The example with some modifications is taken from Ref 3. The problem is one of allocating several types of commercial aircraft to various routes (e.g., New York to Dallas) in order to maximize overall profit (revenue less operational costs) for the system (see Fig. B1). The problem can be stated as follows:

Let m denote the total number of routes

n denote the total number of types of aircraft

a_i denote the anticipated number of passengers on route i per month

b_j denote the number of aircraft of type j

d_{ij} denote the monthly passenger-carrying capacity of aircraft type j on route i

r_i denote the revenue per passenger on route i

s_{ij} denote the monthly operational cost for operating aircraft type j on route i.

Then we seek to find the allocations x_{ij} of aircraft type j to route i for the system that maximize profit subject to constraints on the number of anticipated passengers using the various routes and the available number of each type of aircraft.

Specifically

maximize profit $z(x_{ij})$

Route i	Aircraft type j			
	1	2	3	4
1	1600			900
2	1500	1000	500	1100
3	2800	1400		2200
4	2300	1500	700	1700
5	8100	5700	2900	5500

a. Aircraft Passenger-Carrying Capacity per Month (d_{ij})

Route i	Aircraft type j			
	1	2	3	4
1	18,000			17,000
2	21,000	15,000	10,000	16,000
3	18,000	16,000		17,000
4	16,000	14,000	9,000	15,000
5	10,000	9,000	6,000	1,000

b. Operational Costs in Dollars per Month (s_{ij})

Fig. B1—Sample Data for Aircraft Allocation Problem

Blanks in cell i, j indicate that aircraft type j is never assigned to route i.

$m = 5$ (routes) $n = 4$ (types of aircraft)

Passenger data	Passenger fare data	Aircraft data
$a_1 = 25{,}000$	$r_1 = \$130.00$	$b_1 = 10$
$a_2 = 12{,}000$	$r_2 = \$130.00$	$b_2 = 19$
$a_3 = 18{,}000$	$r_3 = \$\ 70.00$	$b_3 = 25$
$a_4 = 9{,}000$	$r_4 = \$\ 70.00$	$b_4 = 15$
$a_5 = 60{,}000$	$r_5 = \$\ 10.00$	

$$z(x_{ij}) = \underbrace{\sum_{i=1}^{m} r_i \left(\sum_{j=1}^{n} d_{ij} x_{ij} \right)}_{\text{revenue}} - \underbrace{\sum_{i,j}^{m,n} s_{ij} x_{ij}}_{\text{operational costs}} = \sum_{i=1}^{m} \sum_{j=1}^{n} (r_i d_{ij} - s_{ij}) x_{ij} \qquad \text{(B1)}$$

subject to

$$\sum_{j=1}^{n} d_{ij} x_{ij} \le a_i \quad (i = 1, \ldots, m) \quad \text{do not exceed passenger demand for each route } i$$

$$\sum_{i=1}^{m} x_{ij} \le b_j \quad (j = 1, \ldots, n) \quad \text{do not exceed aircraft availability for each type } j$$

Data for $m = 5$, $n = 4$ are given in Table 1 as taken from Ferguson-Dantzig.[3] It should be noted that if $c_{ij} = r_i d_{ij} - s_{ij}$ above, then the form of the problem is that of Prob 3. The Ferguson-Dantzig[3] example requires that all aircraft be allocated, i.e., $\sum_{i=1}^{m} x_{ij} = b_j$ $(j = 1, \ldots, n)$. The algorithm here does not require that all resources be allocated. However, at least all the row or all the column constraints will be binding for an optimal solution. If all the column constraints are binding, then of course the equality constraints on the columns are satisfied.

The example of Ferguson-Dantzig[3] can be formulated from the example here for $m = 5$, $n = 4$ as follows:

Let

$$x_{i5} = a_i - \sum_{j=1}^{4} d_{ij} x_{ij} \qquad \text{(B2)}$$

$$s_{i5} = r_i \qquad d_{i5} = 1$$

then we have

$$\begin{aligned} \max_{x_{ij}} z(x_{ij}) &= \max_{x_{ij}} \left[\sum_{i=1}^{5} r_i (a_i - x_{i5}) - \sum_{i=1}^{5} \left(\sum_{j=1}^{4} s_{ij} x_{ij} \right) \right] \\ &= \max_{x_{ij}} \left[\sum_{i=1}^{5} r_i a_i - \left\{ \sum_{i=1}^{5} \left(\sum_{j=1}^{4} s_{ij} x_{ij} + r_i x_{i5} \right) \right\} \right] \qquad \text{(B3)} \\ &= \sum_{i=1}^{5} r_i a_i - \min_{x_{ij}} \left\{ \sum_{i=1}^{5} \left(\sum_{j=1}^{5} s_{ij} x_{ij} \right) \right\} \end{aligned}$$

subject to

$$\sum_{j=1}^{4} d_{ij} x_{ij} + x_{i5} = \sum_{j=1}^{5} d_{ij} x_{ij} = a_i \qquad (i = 1, \ldots, 4)$$

$$\sum_{i=1}^{4} x_{ij} = b_j \qquad (j = 1, \ldots, 5)$$

Ferguson and Dantzig[3] consider the problem

$$\min_{x_{ij}} \sum_{\substack{i=1 \\ j=1}}^{5,5} s_{ij} x_{ij}$$

subject to the two sets of constraints given.

Details of the solution are printed at each step of the solution for the problem expressed by Eq B1. A completely detailed description illustrating the algorithm on a step-by-step basis is presented between the first and second intermediate printouts (iterations) of phase 2 of the algorithm.

INPUT CONSTANTS

INPUT CONSTANTS FOR OPTIMAL ALLOCATION PROBLEM

MS= 5   NS= 4   FREQ. OF OBJ. FUNCT. PRINTOUT 1

INPUT VALUES FOR CF(I,J)

ROW 1	0.190000E 06	0.	0.	0.100000E 06
ROW 2	0.174000E 06	0.115000E 06	0.550000E 05	0.127000E 06
ROW 3	0.168000E 06	0.820000E 05	0.	0.137000E 06
ROW 4	0.145000E 06	0.910000E 05	0.400000E 05	0.104000E 06
ROW 5	0.710000E 05	0.480000E 05	0.230000E 05	0.450000E 05

INPUT VALUES FOR DE(I,J)

ROW 1	0.160000E 04	0.100000E 01	0.100000E 01	0.900000E 03
ROW 2	0.150000E 04	0.100000E 04	0.500000E 03	0.110000E 04
ROW 3	0.280000E 04	0.140000E 04	0.100000E 01	0.220000E 04
ROW 4	0.230000E 04	0.150000E 04	0.700000E 03	0.170000E 04
ROW 5	0.810000E 04	0.570000E 04	0.290000E 04	0.550000E 04

INPUT VALUES FOR AA(I) OF THE ROW CONSTRAINTS

0.250000E 05   0.120000E 05   0.180000E 05   0.900000E 04

INPUT VALUES FOR BB(J) OF THE COLUMN CONSTRAINTS

0.160000E 02   0.190000E 02   0.250000E 02   0.150000E 02

CR OPTIMAL ALLOCATION PROBLEM

REQ. OF OBJ. FUNCT. PRINTOUT 1   DETAILED PRINTOUT IF ITAB = 1   ITAB= 1

0.	0.100000E 06
0.550000E 05	0.127000E 06
0.	0.137000E 06
0.400000E 05	0.104000E 06
0.230000E 05	0.450000E 05

0.100000E 01	0.900000E 03
0.500000E 03	0.110000E 04
0.100000E 01	0.220000E 04
0.700000E 03	0.170000E 04
0.290000E 04	0.550000E 04

CONSTRAINTS

0.180000E 05	0.900000E 04	0.600000E 05

UMN CONSTRAINTS

0.250000E 02	0.150000E 02

B

RAC

PHASE 1, ITERATIONS

```
MATRIX MAXIMUM ITERATION NUMBER      1
     ALLOCATION SELECTED     ROW     1     COL.     1     XX(I,J)=   0.10000E 02
     VALUE OF OBJECTIVE FUNCTION =  0.19000E 07

MATRIX MAXIMUM ITERATION NUMBER      2
     ALLOCATION SELECTED     ROW     3     COL.     4     XX(I,J)=   0.81818E 01
     VALUE OF OBJECTIVE FUNCTION =  0.30209E 07

MATRIX MAXIMUM ITERATION NUMBER      3
     ALLOCATION SELECTED     ROW     2     COL.     4     XX(I,J)=   0.68182E 01
     VALUE OF OBJECTIVE FUNCTION =  0.38868E 07

MATRIX MAXIMUM ITERATION NUMBER      4
     ALLOCATION SELECTED     ROW     2     COL.     2     XX(I,J)=   0.45000E 01
     VALUE OF OBJECTIVE FUNCTION =  0.44043E 07

MATRIX MAXIMUM ITERATION NUMBER      5
     ALLOCATION SELECTED     ROW     4     COL.     2     XX(I,J)=   0.60000E 01
     VALUE OF OBJECTIVE FUNCTION =  0.49503E 07

MATRIX MAXIMUM ITERATION NUMBER      6
     ALLOCATION SELECTED     ROW     5     COL.     2     XX(I,J)=   0.85000E 01
     VALUE OF OBJECTIVE FUNCTION =  0.53583E 07

MATRIX MAXIMUM ITERATION NUMBER      7
     ALLOCATION SELECTED     ROW     5     COL.     3     XX(I,J)=   0.39828E 01
     VALUE OF OBJECTIVE FUNCTION =  0.54499E 07
     ITERATION     1         PRIOR VALUE OF OBJ FUNCTION  0.54499E 07        MAX. VIOLATION OF
```


RAC

I,J)= 0.10000E 02

I,J)= 0.81818E 01

I,J)= 0.68182E 01

I,J)= 0.45000E 01

I,J)= 0.60000E 01

I,J)= 0.85000E 01

I,J)= 0.39828E 01

).54499E 07   MAX. VIOLATION OF DUAL CONSTR. -0.96428E 05

B

```
                              DETAILED INTERMEDIATE PRINTOUT

                 PRIOR VALUES OF THE DUAL VARIABLES U(I) IN ORDER
                   0.            0.112207E 03    0.606489E 02    0.588046E 02

                 PRIOR VALUES OF THE DUAL VARIABLES V(J) IN ORDER
                   0.190000E 06  0.279310E 04    0.              0.357242E 04
        ENTERING VECTOR    1    4       LEAVING VECTOR    2    4       XX(IENTER,

           CURRENT SOLUTION ARRAY XX(I,J)

          ROW  1
                   0.100000E 02  0.              0.              0.681818E 01
          ROW  2
                   0.            0.120000E 02    0.              0.
          ROW  3
                   0.            0.              0.              0.818182E 01
          ROW  4
                   0.            0.600000E 01    0.              0.
          ROW  5
                   0.            0.100000E 01    0.187241E 02    0.

                UNUSED RESOURCES      COLUMNS 1 THRU NS IN ORDER

                   0.            0.              0.627586E 01    0.

                UNUSED CAPACITIES      ROWS 1 THRU MS IN ORDER

                   0.286364E 04  0.              0.              0.
        ITERATION     2          PRIOR VALUE OF OBJ FUNCTION  0.61074E 07        MAX. V
```

```
ILED INTERMEDIATE PRINTOUT

AL VARIABLES U(I) IN ORDER
22U7E 03     0.606489E 02    0.588046E 02    0.793103E 01

AL VARIABLES V(J) IN ORDER
9310E 04     0.              0.357242E 04
EAVING VECTOR    2    4        XX(IENTER,JENTER) =  0.68182E 01

)

             0.              0.681818E 01

00C0E 02     0.              0.

             0.              0.818182E 01

0000E 01     0.              0.

0C00E 01     0.187241E 02    0.
UMNS 1 THRU NS IN ORDER

             0.627586E 01    0.
NS 1 THRU MS IN ORDER

             0.              0.              0.
OF OBJ FUNCTION  0.61074E 07       MAX. VIOLATION OF DUAL CONSTR. -0.55661E 05
```


B

RAC

DETAILED DESCRIPTION OF ONE ITERATION OF PHASE 2

A detailed numerical examination of one iteration of phase 2 of the algorithm is presented here. Given the solution array XX(I,J) of the previous page, we wish to test optimality of the solution. Proceeding as in step 2 of the algorithm we solve Eqs 15 for $(u_1, \ldots, u_5, v_1, \ldots, v_4)$.

Since $x_{63} = 6.27586 > 0$ and $x_{51} = 2863.63 > 0$, we have $v_3 = 0$ and $u_1 = 0$ immediately, i.e., we solve for the zero-valued dual-space variables first. The remaining seven equations

$$d_{ij}u_i + v_j = c_{ij} \quad \text{if} \quad x_{ij} > 0$$

are solved sequentially. We have

$$x_{11} = 10 > 0 \Rightarrow v_1 = c_{11} = 190{,}000$$

$$x_{14} = 6.81818 > 0 \Rightarrow v_4 = c_{14} = 100{,}000$$

$$x_{53} = 18.7241 > 0 \Rightarrow u_5 = \frac{c_{53}}{d_{53}} = \frac{23{,}000}{2{,}900} = 7.93103$$

$$x_{52} = 1 > 0 \Rightarrow v_2 = c_{52} - d_{52}u_5 = 48{,}000 - (5700)(7.93103) = 2793.10$$

$$x_{22} = 12 > 0 \Rightarrow u_2 = \frac{c_{22} - v_2}{d_{22}} = \frac{115{,}000 - 2793.10}{1000} = 112.207$$

$$x_{42} = 6 > 0 \Rightarrow u_4 = \frac{c_{42} - v_2}{d_{42}} = \frac{91{,}000 - 2793.10}{1500} = 58.8046$$

$$x_{34} = 7.54545 > 0 \Rightarrow u_3 = \frac{c_{34} - v_4}{d_{34}} = \frac{137{,}000 - 100{,}000}{2200} = 16.8182$$

Moving on to step 3 of the algorithm the optimality test is now made (i.e., Conditions 7$\bar{\text{D}}$ in scalar form of the $m + n + mn = 29$ equalities of Eq 16 are tested). It is seen directly that u_i $(i = 1, \ldots, 5)$ and v_j $(j = 1, \ldots, 4)$ are nonnegative. Seven of the remaining $mn = 20$ inequalities; namely, $w_{s_{ij}} = d_{ij}u_i + v_j - c_{ij} = 0$ for $x_{ij} > 0$ (i.e., for x_{11}, x_{14}, x_{53}, x_{52}, x_{22}, x_{42}, and x_{34}) are equalities, hence only the remaining $20 - 7 = 13$ values of $w_{s_{ij}}$ are tested for nonnegativity. Computing directly we have

$$w_{s_{12}} = d_{12}u_1 + v_2 - c_{12} = (1)(0) + 2793.10 - 0 > 0$$

$$w_{s_{13}} = d_{13}u_1 + v_3 - c_{13} = (1)(0) + 0 - 0 = 0$$

$$w_{s_{21}} = d_{21}u_2 + v_1 - c_{21} = (1500)(112.207) + 190{,}000 - 174{,}000 > 0$$

$$w_{s_{23}} = d_{23}u_2 + v_3 - c_{23} = (500)(112.207) + 0 - 55{,}000 > 0$$

$$w_{s_{24}} = d_{24}u_2 + v_4 - c_{24} = (1100)(112.207) + 100{,}000 - 127{,}000 > 0$$

$$w_{s_{31}} = d_{31}u_3 + v_1 - c_{31} = (2800)(16.8182) + 190{,}000 - 168{,}000 > 0$$

$$w_{s_{32}} = d_{32}u_3 + v_2 - c_{32} = (1400)(16.8182) + 2793.10 - 82{,}000 = -55{,}661.42 < 0$$

$$w_{s_{33}} = d_{33}u_3 + v_3 - c_{33} = (1)(16.8182) + 0 - 0 > 0$$

$$w_{s_{41}} = d_{41}u_4 + v_1 - c_{41} = (2300)(58.8046) + 190{,}000 - 145{,}000 > 0$$

$$w_{s_{43}} = d_{43}u_4 + v_3 - c_{43} = (700)(58.8046) + 0 - 40{,}000 = 41{,}163.22 - 40{,}000 > 0$$

$$w_{s_{44}} = d_{44}u_4 + v_4 - c_{44} = (1700)(58.8046) + 100{,}000 - 104{,}000 > 0$$

$$w_{s_{51}} = d_{51}u_5 + v_1 - c_{51} = (8100)(7.93103) + 190{,}000 - 71{,}000 > 0$$

$$w_{s_{54}} = d_{54}u_5 + v_4 - c_{54} = (5500)(7.93103) + 100{,}000 - 45{,}000 > 0$$

It is seen at this point that the only violation and hence the maximum violation (step 4 of the algorithm) of Conditions $7\bar{D}$ of Eq 16 is

$$w_{s_{32}} = -55661.42 < 0$$

Hence the solution is not optimal and the vector $d_{32}\vec{e}_3 + \vec{e}_7$ enters the basis.

The vector selected to leave the basis is determined next, but first it is necessary to determine the components y_{ij}^{32} of the entering vector $d_{32}\vec{e}_3 + \vec{e}_7$ in terms of the current basis. The set of $m + n = 5 + 4 = 9$ linear equations Eq 21 is thus solved (step 5 of the algorithm). We have for the case at hand the following system:

$$y_{11}^{32}(1600) + y_{14}^{32}(900) + y_{15}^{32}(1) = 0$$

$$y_{22}^{32}(1000) = 0$$

$$y_{34}^{32}(2200) = d_{32} = 1400$$

$$y_{42}^{32}(1500) = 0$$

$$y_{52}^{32}(5700) + y_{53}^{32}(2900) = 0$$

$$y_{11}^{32} = 0$$

$$y_{22}^{32} + y_{42}^{32} + y_{52}^{32} = 1$$

$$y_{53}^{32} + y_{63}^{32} = 0$$

$$y_{14}^{32} + y_{34}^{32} = 0$$

The solution to the foregoing equations is

$$y_{22}^{32} = 0,\ y_{34}^{32} = \frac{1400}{2200},\ y_{42}^{32} = 0,\ y_{52}^{32} = 1,\ y_{11}^{32} = 0,$$

$$y_{53}^{32} = -\frac{5700}{2900},\ y_{63}^{32} = \frac{5700}{2900},\ y_{14}^{32} = -\frac{1400}{2200},\ y_{15}^{32} = \frac{(-900)(-1400)}{2200}$$

Thus the entering vector $d_{32}\vec{e}_3 + \vec{e}_7 = 1400\vec{e}_3 + \vec{e}_7$ can be written as the following linear combination of the current basis vectors.

$$1400\vec{e}_3 + \vec{e}_7 = \begin{cases} -\frac{1400}{2200}[900\vec{e}_1 + \vec{e}_9] + \frac{(-900)(-1400)}{2200}[\vec{e}_1] \\ +\frac{1400}{2200}[2200\vec{e}_3 + \vec{e}_9] + 1[5700\vec{e}_5 + \vec{e}_7] \\ -\frac{5700}{2900}[2900\vec{e}_5 + \vec{e}_8] + \frac{5700}{2900}[\vec{e}_8] \end{cases}$$

The vector now selected to leave the basis is determined from the indexes that yield the minimum in the brackets. Using Eq 29 (step 6 of the algorithm) we have

$$\hat{\theta} = \min\left\{\frac{8.18182}{\frac{1400}{2200}}, \frac{1}{1}, \frac{6.27586}{\left(\frac{5700}{2900}\right)}, \frac{2863.64}{\frac{(900)(1400)}{2200}}\right\}$$

$$= 1$$

and the minimizing indexes are (5,2), since $\frac{x_{52}}{y_{52}^{32}} = \frac{1}{1}$ yields the minimum value $\hat{\theta}$. Thus the vector $d_{52}\vec{e}_5 + \vec{e}_7 = 5700\vec{e}_5 + \vec{e}_7$ leaves the basis.

The new solution is evaluated (step 7) using Eqs 28 and appears in the next intermediate printout. The algorithm again returns to step 2 for the next iteration. The value of the objective function was increased by $\hat{\theta}(-u_{s_{32}})$ = \$55,661.42 during this iteration.

RAC

PHASE 2, REMAINING ITERATIONS

```
                                        DETAILED INTERMEDIATE PRINTOUT

                              PRIOR VALUES OF THE DUAL VARIABLES U(I) IN ORDER
                              C.              0.112207E 03    0.168182E 02    0.588046
                              PRIOR VALUES OF THE DUAL VARIABLES V(J) IN ORDER
                              C.190000E 06    C.279310E 04    0.              0.100000
      ENTERING VECTOR     3    2           LEAVING VECTOR     5    2           XX(IE

            CURRENT SOLUTION ARRAY XX(I,J)

         ROW  1
                        C.100000E 02    0.              0.              0.745455E
         ROW  2
                        C.              C.120000E 02    C.              0.
         ROW  3
                        C.              C.100000E 01    0.              0.754545E
         ROW  4
                        C.              C.600000E 01    0.              0.
         ROW  5
                        0.              0.              0.206897E 02    0.
                  UNUSED RESOURCES       COLUMNS 1 THRU NS IN ORDER

                        C.              C.              C.431034E 01    0.
                  UNUSED CAPACITIES      ROWS 1 THRU MS IN ORDER

                        0.229091E 04    C.              0.              0.
      ITERATION     3          PRIOR VALUE OF OBJ FUNCTION  0.61630E 07         M
```

```
TAILED INTERMEDIATE PRINTOUT

DUAL VARIABLES U(I) IN ORDER
.122207E 03     0.168182E 02     0.588046E 02     0.793103E 01

DUAL VARIABLES V(J) IN ORDER
.793110E 04     0.               0.100000E 06
LEAVING VECTOR     5     2          XX(IENTER,JENTER) =   0.10000E 01

J)

                0.               0.745455E 01

200000E 02      0.               0.

000000E 01      0.               0.754545E 01

000000E 01      0.               0.

                0.206897E 02     0.

LUMNS 1 THRU NS IN ORDER

                0.431034E 01     0.

WS 1 THRU MS IN ORDER

                0.               0.               0.
CF OBJ FUNCTION   0.61630E 07        MAX. VIOLATION OF DUAL CONSTR. -0.26727E 05
```


B

RAC

DETAILED INTERMEDIATE PRINTOUT

PRIOR VALUES OF THE DUAL VARIABLES U(I) IN ORDER

0.	0.565455E 02	0.168182E 02	0.216970E 02	0.793

PRIOR VALUES OF THE DUAL VARIABLES V(J) IN ORDER

0.190000E 06	0.584545E 05	0.	0.100000E 06

ENTERING VECTOR 2 3 LEAVING VECTOR 6 3 XX(IENTER,JENTER)

CURRENT SOLUTION ARRAY XX(I,J)

ROW 1	0.100000E 02	0.	0.	0.882602E 01
ROW 2	0.	0.984483E 01	0.431034E 01	0.
ROW 3	0.	0.315517E 01	0.	0.617398E 01
ROW 4	0.	0.600000E 01	0.	0.
ROW 5	0.	0.	0.206897E 02	0.

UNUSED RESOURCES COLUMNS 1 THRU NS IN ORDER

0.	0.	0.	0.

UNUSED CAPACITIES ROWS 1 THRU MS IN ORDER

0.105658E 04	0.	0.	0.	0.

ITERATION 4 PRIOR VALUE OF OBJ FUNCTION 0.62782E 07 MAX. VIOLATIO

```
PRINTOUT

IN ORDER
82E 02     0.216970E 02     0.793103E 01

IN ORDER
           0.100000E C6
   3          XX(IENTER,JENTER) =   0.43103E 01


           0.882602E 01

34E 01     0.

           0.617398E 01

           0.

97E 02     0.

ORCER

           0.

DER

           0.                0.
.62782E 07         MAX. VIOLATION OF DUAL CONSTR. -0.12853E 01
```


B

DETAILED INTERMEDIATE PRINTOUT

PRIOR VALUES OF THE DUAL VARIABLES U(I) IN ORDER
0. 0.565455E 02 0.168182E 02 0.21697

PRIOR VALUES OF THE DUAL VARIABLES V(J) IN ORDER
0.190000E 06 0.584545E 05 0.267273E 05 0.10000
ENTERING VECTOR 5 5 LEAVING VECTOR 1 5 XX(I

CURRENT SOLUTION ARRAY XX(I,J)

ROW 1				
	0.100000E 02	0.	0.	0.10000
ROW 2				
	0.	0.800000E 01	0.800000E 01	0.
ROW 3				
	0.	0.500000E 01	0.	0.50000
ROW 4				
	0.	0.600000E 01	0.	0.
ROW 5				
	0.	0.	0.170000E 02	0.

UNUSED RESOURCES COLUMNS 1 THRU NS IN ORDER

0. 0. 0. 0.

UNUSED CAPACITIES ROWS 1 THRU MS IN ORDER

0. 0. 0. 0.

TOTAL EXECUTION TIME FOR ALGORITHM = 0.7333333 SEC.

NUMBER OF ITERATIONS AFTER INITIAL SOLUTION 4

MAXIMUM VIOLATION OF DUAL CONSTRAINTS 0.

ILED INTERMEDIATE PRINTOUT

AL VARIABLES U(I) IN ORDER
5455E 02   0.168182E 02   0.216970E 02   -0.128527E 01

AL VARIABLES V(J) IN ORDER
4545E 05   0.267273E 05   0.100000E 06
EAVING VECTOR   1   5   XX(IENTER,JENTER' =   0.10700E 05

0

0.   0.100000E 02

0000E 01   0.800000E 01   0.

0000E 01   0.   0.500000E 01

0000E 01   0.   0.

0.170000E 02   0.

UMNS 1 THRU NS IN ORDER

0.   0.

S 1 THRU MS IN ORDER

0.   0.   0.107000E 05

0.7333333 SEC.

ION   4

0.

B

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DETAILS OF THE SOLUTION

```
ROW  1    COLUMN  1    ALLOCATION  0.10000E 02    RETURN FROM ALLOCATION
ROW  1    COLUMN  4    ALLOCATION  0.10000E 02    RETURN FROM ALLOCATION
ROW  2    COLUMN  2    ALLOCATION  0.80000E 01    RETURN FROM ALLOCATION
ROW  2    COLUMN  3    ALLOCATION  0.80000E 01    RETURN FROM ALLOCATION
ROW  3    COLUMN  2    ALLOCATION  0.50000E 01    RETURN FROM ALLOCATION
ROW  3    COLUMN  4    ALLOCATION  0.50000E 01    RETURN FROM ALLOCATION
ROW  4    COLUMN  2    ALLOCATION  0.60000E 01    RETURN FROM ALLOCATION
ROW  5    COLUMN  3    ALLOCATION  0.17000E 02    RETURN FROM ALLOCATION

VALUE OF PRIMAL OBJECTIVE FUNCTION        0.629200E 07

VALUES OF  THE DUAL SPACE VARIABLES (LAGRANGE MULTIPLIERS , SHADOW PRICES)
        ROW  1    U(I)=  0.13[illegible]16E 02
        ROW  2    U(I)=  0.64000E 02
        ROW  3    U(I)=  0.22143E 02
        ROW  4    U(I)=  0.26667E 02
        ROW  5    U(I)=  0.
     COLUMN  1    V(J)=  0.16917E 06
     COLUMN  2    V(J)=  0.51000E 05
     COLUMN  3    V(J)=  0.23000E 05
     COLUMN  4    V(J)=  0.88286E 05

VALUE OF DUAL OBJECTIVE FUNCTION        0.629200E 07
```

N FROM ALLOCATION 0.19000E 07
N FROM ALLOCATION 0.10000E 07
N FROM ALLOCATION 0.92000E 06
N FROM ALLOCATION 0.44000E 06
N FROM ALLOCATION 0.41000E 06
N FROM ALLOCATION 0.68500E 06
N FROM ALLOCATION 0.54600E 06
N FROM ALLOCATION 0.39100E 06

RICES)

B

Appendix C

COMPARATIVE SIMPLEX ALGORITHM

This appendix contains a general simplex algorithm programmed specifically to solve Prob 3. The program accepts the same input data as the MATMAX algorithm with the exception of the IWRITE and ITAB information used in the MATMAX algorithm. The algorithm here has been used for comparative purposes, with particular attention given to the times required by the two algorithms to solve the same problem.

```
ISN        SOURCE STATEMENT

  0 $IBFTC MATLIN
  1          DIMENSION CE(20,24),DE(21,25),AA(20),BB(24)
  2          DIMENSION AS(42,474),BS(45),CS(525)
  3          DIMENSION  IPATHS(45),CTS(45),BTS(45),ATSI(45)
  4          DIMENSION ZS(525),ZMCS(525),ATSJ(525)
  5  18001 FORMAT(1H1)
  6  18002 FORMAT(1H )
  7  18003 FORMAT(3I10)
 10  18004 FORMAT(6F12.6)
 11  18005 FORMAT(15X,6E16.6)
 12  18006 FORMAT(8X,2HMS,8X,2HNS,2X,15HPRINT FREQUENCY)
 13  18007 FORMAT(30X,46HINPUT CONSTANTS FOR OPTIMAL ALLOCATION PROBLEM//)
 14  18008 FORMAT(15X,24HINPUT VALUES FOR CE(I,J)//)
 15  18009 FORMAT(15X,24HINPUT VALUES FOR DE(I,J)//)
 16  18010 FORMAT(15X,45HINPUT VALUES FOR AA(I) OF THE ROW CONSTRAINTS//)
 17  18011 FORMAT(15X,48HINPUT VALUES FOR BB(J) OF THE COLUMN CONSTRAINTS//)
 20  18015 FORMAT(//37HTOTAL EXECUTION TIME FOR ALGORITHM = ,F12.7,1X,4HSEC.)
 21  18019 FORMAT(//34HVALUE OF PRIMAL OBJECTIVE FUNCTION,E20.6)
 22  18027 FORMAT(6X,4H ROW,I3)
 23  18020 FORMAT(10I10)
 24  18028 FORMAT(6X,7H COLUMN,I3)
 25        WRITE(6,18001)
 26        WRITE(6,18007)
 27        READ(5,18003)MM,NN
 32        WRITE(6,18006)
 33        WRITE(6,18003)MM,NN
 34        WRITE(6,18002)
 35        WRITE(6,18008)
 36        DO 18035 I=1,MM
 37        READ(5,18004)(CE(I,J),J=1,NN)
 44        WRITE(6,18027)I
 45  18035 WRITE(6,18005)(CE(I,J),J=1,NN)
 53        WRITE(6,18002)
 54        WRITE(6,18009)
 55        DO 18040 I=1,MM
 56        READ(5,18004)(DE(I,J),J=1,NN)
 63        WRITE(6,18027)I
 64  18040 WRITE(6,18005)(DE(I,J),J=1,NN)
 72        WRITE(6,18002)
 73        WRITE(6,18010)
 74        READ(5,18004)(AA(I),I=1,MM)
101        WRITE(6,18005)(AA(I),I=1,MM)
106        WRITE(6,18002)
107        WRITE(6,18011)
110        READ(5,18004)(BB(J),J=1,NN)
115        WRITE(6,18005)(BB(J),J=1,NN)
122        WRITE(6,18001)
123        MS = MM + NN
124        NS = MM * NN
125        DO 18110 I = 1,MM
126        DO 18110 J = 1,NN
127        K = NN *(I - 1) + J
130        CS(K) =-CE(I,J)
131  18110 AS(I,K) = DE(I,J)
134        DO 18120 I = 1,NN
```

```
ISN         SOURCE STATEMENT

135         DO 18120 J = 1,MM
136         II = MM + I
137         JJ = NN * (J - 1) + I
140  18120  AS(II,JJ) = 1.
143         DO 18130 I = 1,MM
144  18130  BS(I) = AA(I)
146         DO 18140 J = 1,NN
147         JJ = MM + J
150  18140  BS(JJ) = BB(J)
152         DO 18150 I = 1,MS
153         J = I + NS
154         IPATHS(I) = J
155  18150  AS(I,J) = 1.
157         NNS = MS + NS
     C --- SIMPLEX METHOD SOLUTION OF LINEAR PROGRAMMING PROBLEM
160         EPSLP = .1E-5
     C
161          CALL TODAY(0,ITIME,IDAT)
     C
     C --- BEGIN ITERATION
     C
162  18200  COST=0.
163         DO 18205 I=1,MS
164  18205  COST = COST + CTS(I)*BS(I)
     C
     C --- COMPUTE ZS AND ZMCS VECTORS
     C
166  18216  DO 18220 J=1,NNS
167         ZTS=0.
170         DO 18217 I=1,MS
171  18217  ZTS=ZTS + CTS(I)*AS(I,J)
173         ZS(J)=ZTS
174  18220  ZMCS(J)=ZS(J)-CS(J)
     C
     C --- SELECT MAXIMUM ZMCS.  IF NO POSITIVE, END.
     C
176  18230  CMAX=ZMCS(1)
177         JMAX=1
200         DO 18250 J=2,NNS
201         IF(CMAX-ZMCS(J))18240,18250,18250
202  18240  CMAX=ZMCS(J)
203         JMAX=J
204  18250  CONTINUE
206         IF(ZMCS(JMAX)-EPSLP)18800,18800,18260
     C
     C --- SELECT MINIMUM DS(I)=BS(I)/AS(I,JMAX) WHERE A(I,JMAX) IS POSITIVE
     C
207  18260  DSMIN=1.E+35
210  18270  DO 18350 I=1,MS
211         IF(AS(I,JMAX)-.1E-6)18350,18350,18300
212  18300  DST=BS(I)/AS(I,JMAX)
213         IF(DST-DSMIN)18310,18350,18350
214  18310  DSMIN=DST
215         IMIN=I
216  18350  CONTINUE
```

```
 ISN         SOURCE STATEMENT

      C
      C --- COMPUTE NEW MATRIX ATS
      C
 220          DO 18400 I=1,MS
 221          BTS(I)=AS(I,JMAX)    *    BS(IMIN)    /        AS(IMIN,JMAX)
 222  18400   ATSI(I)=AS(I,JMAX)/AS(IMIN,JMAX)
 224          TEMP=AS(IMIN,JMAX)
 225          TEMP2 = ZMCS(JMAX) / TEMP
 226          THETA = BS(IMIN) / AS(IMIN,JMAX)
 227          COST = COST - THETA * ZMCS(JMAX)
 230          WRITE(6,18019)COST
 231          DO 18410 J = 1,NNS
 232  18410   ATSJ(J)=AS(IMIN,J)
 234          DO 18525 I=1,MS
 235          IF(I-IMIN)18450,18500,18450
 236  18450   IF(ATSI(I))18455,18525,18455
 237  18455   BS(I)=BS(I)-BTS(I)
 240          DO 18475 J = 1,NNS
 241          IF(ATSJ(J))18460,18475,18460
 242  18460   AS(I,J) = AS(I,J) - ATSI(I) * ATSJ(J)
 243          IF(AS(I,J)-.1E-10)18462,18462,18475
 244  18462   IF(AS(I,J) - .1E-10)18475,18465,18465
 245  18465   AS(I,J)=0.
 246  18475   CONTINUE
 250          GO TO 18525
 251  18500   DO 18510 J=1,NNS
 252          AS(I,J)=AS(I,J)/TEMP
 253          IF(AS(I,J)-.1E-10)18502,18502,18510
 254  18502   IF(AS(I,J) - .1E-10)18510,18505,18505
 255  18505   AS(I,J)=0.
 256  18510   CONTINUE
 260          BS(I)=BS(I)/TEMP
 261  18525   CONTINUE
      C
 263          DO 18550 J=1,NNS
 264  18550   ZMCS(J) = ZMCS(J) - ATSJ(J) *TEMP2
      C --- SUBSTITUTE IPATH OF JMAX FOR IMIN, C OF JMAX FOR IMIN
      C
 266          IPATHS(IMIN)=JMAX
 267          CTS(IMIN)=CS(JMAX)
      C
      C --- TRANSFER BACK TO BEGIN ITERATION
      C
 270          GO TO 18230
      C
 271  18800   CONTINUE
 272           CALL TODAY(1,ITIME,IDAT)
 273           TIME = FLOAT(ITIME)/60.
 274          WRITE(6,18015)TIME
 275          WRITE(6,18020)(IPATHS(I),I=1,MS)
 302          WRITE(6,18005)(BS(I),I=1,MS)
 307          WRITE(6,18005)(CTS(I),I=1,MS)
 314          CALL EXIT
 315          END
```


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13. ABSTRACT The algorithm described in this paper is used to solve a special class of linear programming problems characterized by constraint coefficient matrices having generalized transportation structure. Specifically, n available resources are allocated to m capacity-limited operations (where the operational capability of assigning the jth resource to the ith operation is known) such as to maximize the total profit for the system. The row-and-column structure of such problems permits an algorithm more efficient than the general simplex algorithm to be used to solve moderate-sized problems (problems where loop-tracing techniques or equivalent schemes are not required). It is not required in the problem statement that all the resources be allocated or that all operations be performed to capacity limits. It is characteristic of such problems, however, that the optimizing solution usually requires that at least one of the two conditions holds, i.e., either supply or demand is exhausted. The paper contains a description of the algorithm, a computer program, an example illustrating its application, and some comparisons with the general simplex algorithm in solving the same problem. ( )

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